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Ellipsoidal Coordinates—A Natural Coordinate System for Calculations of Laser Irradiations of Slabs*

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In most focused laser irradiations of slabs there are several significant two- and three-dimensional effects in the resultant material motion. By using a "natural" coordinate system it is possible to account for the most significant features of these effects in a one-dimensional calculation. The essential requirement is to find a coordinate system wherein the contours of constant density, temperature, etc., coincide approximately with the coordinate lines. For one such coordinate system, an "ellipsoidal" coordinate system, we obtain the fluid equations and discuss the approximations that are necessary to obtain one-dimensional equations.

1. INTRODUCTION

In many instances plasmas have been created in the laboratory by focusing intense laser pulses on a solid surface. In the immediate vicinity of the center of the focal spot the consequent plasma motion is planar. For this reason, one-dimensional planar geometry has often been used to study the plasma motion. However, as soon as the plasma expands outward to distances on the order of the focal spot size, there begins to be a significant amount of lateral expansion and the assumption of planarity breaks down. Indeed, for distances large compared to the focal spot size, the expansion becomes approximately spherical.

Clearly, we are faced with solving a problem which is intrinsically two dimensional. However, two dimensional calculations present several problems. Resolution in two dimensions comparable to what can readily be obtained in one dimension requires large amounts of computer time and storage space. By suitably tailoring the coordinate system, significant reductions of these requirements appear possible without appreciable sacrifice in accuracy. A trivial example of this is a problem which has spherical symmetry. Using a spherical coordinate system, the variation is only in the radial direction, resulting in a one-dimensional problem.

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The essential feature of spherically symmetric problems which permits a one-dimensional treatment is that there is no variation in the θ or φ directions. In cases of approximate spherical symmetry, the angular dependence can be neglected as a first approximation. Thus, the key to "one-dimensionalizing" a problem is the use of a coordinate system wherein the variation with respect to two of the three coordinates is negligible.

A typical problem is sketched in Fig. 1. It consists of a laser beam focused on the

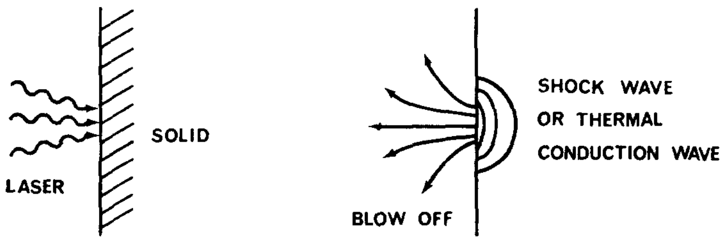

FIG. 1. Sketch of a typical laser irradiation of a solid surface.

surface of a slab with a focal spot radius r_0 (Fig. 1a). As a result, a shock wave or a thermal conduction wave propagates into the slab and surface material blows off into the vacuum (Fig. 1b). A one-dimensional planar calculation effectively restricts the direction of flow to be normal to the slab surface, whereas there is in fact considerable lateral flow. The contours of contant temperature and pressure are nearly elliptic; the lines of mass and energy flow (streamlines) are nearly orthogonal to the contours and are approximately hyperbolic. A coordinate system which has similar features is, in three dimensions, the oblate ellipsoidal coordinate system. [1, 2] The coordinate grid in a plane containing the symmetry axis is shown in Fig. 2. The grid consists of confocal ellipses and confocal hyperbolas. If we use a coordinate system such as this one, then as a first approximation we should be able to neglect the variation of temperature, pressure, etc., along the ellipses. As a result, the problem becomes one-dimensional since the various quantities of interest are functions of only one coordinate.

At this stage in the development of ellipsoidal hydrodynamics, we recognize that one-dimensional ellipsoidal calculations should not be regarded as a complete substitute for full two-dimensional calculations, but rather as a complementary technique. For example, in parameter studies, many one-dimensional ellipsoidal calculations can be made, using relatively little computer time, followed by a few two dimensional calculations for cases of particular interest.

The equations relating the ellipsoidal coordinates to the cylindrical coordinates r and z are obtained in Section 2. The one-dimensionalized hydrodynamic equations are then obtained in Section 3. In Section 4 we discuss briefly how our treatment

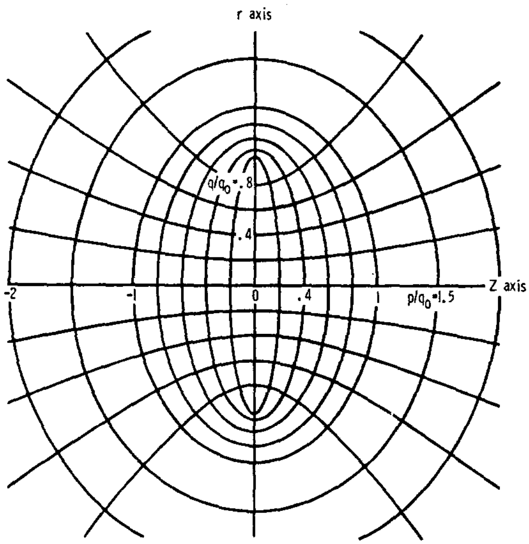

FIG. 2. Ellipsoidal coordinate system (in $r - z$ plane). The grid consists of confocal ellipses and confocal hyperbolas.

relates to other treatments of two-dimensional effects. Section 5 describes a sample laser problem, comparing the results obtained with Cartesian coordinates and with ellipsoidal coordinates. In Appendix A we obtain the hydrodynamic equations in orthogonal curvilinear coordinates and then specialize the results to ellipsoidal coordinates. In Appendix B the difference equations used for the sample calculations are given.

2. ELLIPSOIDAL COORDINATES

The laser irradiations with which we are concerned here are generally assumed to be symmetric about the axis of the laser beam, which we also assume to be perpendicular to the solid surface. Thus, we define this axis to be the z axis of cylindrical coordinates so that all the quantities of interest are independent of ϕ, the angle around this axis. The $z = 0$ plane is defined as the initial surface of the solid. Ellipsoidal coordinates have been used in several areas of physics for some time (see [1, 2] for example). We use the same geometry, of course, but a somewhat different form of the coordinate variables. In this section we briefly review the essential points of the ellipsoidal coordinate system. The equation [3]

$$\frac{r^2}{p^2 + q_0^2} + \frac{z^2}{p^2} = 1 \tag{1}$$

describes an ellipse, which has axes coincident with the r and z axes, has a focus on the r axis at $r = q_0$, and intersects the z axis at $z = p$. Similarly, the equation

$$\frac{r^2}{q^2} - \frac{z^2}{q_0{}^2 - q^2} = 1 \tag{2}$$

describes a hyperbola which has a focus on the r axis at $r = q_0$ and intersects the r axis at $r = q$. The family of ellipses obtained by varying p (and holding q_0 fixed) is everywhere orthogonal to the family of hyperbolas obtained by varying $q(0 < q < q_0)$. Consequently, we may take p and q as the coordinates of our ellipsoidal coordinates. Oblate ellipsoids are obtained by rotating the ellipses around the z axis, which is now also the p axis.

From Eqs. (1) and (2) we may obtain r and z as functions of p and q:

$$r = q(1 + p^2/q_0{}^2)^{1/2}, \tag{3}$$

$$z = p(1 - q^2/q_0{}^2)^{1/2}. \tag{4}$$

From Eq. (4) it can be seen that we take p to have the same sign as z. As a consequence, moving across the r axis for $r > q_0$ produces a discontinuous change in p. However, this will not cause us any particular problem. Near the origin, the coordinate system is essentially planar, with $q \approx r$ and $p \approx z$, while at large distances from the origin ($|p| \gg q_0 \geqslant q$) it becomes essentially spherical with $p^2 \approx r^2 + z^2$. The ellipsoidal coordinate system was originally arrived at by looking for a coordinate system whose grid coincided approximately with the contours of constant temperature, etc. It should be emphasized, however, that the coordinate system itself is stationary; it does not move with the material.

3. Ellipsoidal Hydrodynamics

The basic idea of using ellipsoidal coordinates is, of course, applicable to many types of analytic or machine calculations concerning laser irradiated slabs. In this paper, we will be concerned only with fluid problems. The equations with which we will be concerned are given by

$$(\partial\rho/\partial t) + \nabla \cdot (\rho\mathbf{v}) = 0, \tag{5}$$

$$\rho(\partial\mathbf{v}/\partial t) + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla P = 0 \tag{6}$$

$$(\partial/\partial t)(e + \tfrac{1}{2}v^2) + \mathbf{v} \cdot \nabla(e + \tfrac{1}{2}v^2) + (1/\rho)\nabla \cdot (P\mathbf{v}) = \dot{Q}, \tag{7}$$

where ρ is the density, $\mathbf{v}$ is the velocity, P is the pressure, e is the internal (thermal) energy per unit mass, and $\dot{Q}$ is a sum of "source" terms (power per unit mass).

The principal source terms of interest here are the heat conduction and laser deposition terms,

$$\rho \dot{Q}_C = \nabla \cdot (K \nabla T), \tag{8}$$

$$\rho \dot{Q}_L = -\nabla \cdot \mathbf{\Phi}, \tag{9}$$

where K is the thermal conductivity, T is the temperature, and $\mathbf{\Phi}$ is the laser energy flux vector.

In Appendix A, Eqs. (5)–(9) are transformed into ellipsoidal coordinates. In accordance with our approximation we assume that P, T, $\dot{p}$, etc., are independent of q, and $\dot{q} = 0$. By $\dot{p}$ we mean $dp(t)/dt$ where $p(t)$ is the coordinate of a point moving with the fluid. We will also assume that $\Phi_q = 0$ so that the laser light propagates along the hyperbolas. Except for $q \approx q_0$ this is a reasonable description of the focal region. In effect we are assuming that the dominant energy and mass flows are in the p direction, along the hyperbolas of constant q, and we are ignoring the flows in the q direction. The equations then become

$$\frac{\partial \rho}{\partial t} + \dot{p}\frac{\partial \rho}{\partial p} \quad \frac{\rho}{p^2 + q_0^2 - q^2}\frac{\partial}{\partial p}[\dot{p}(p^2 + q_0^2 - q^2)] = 0, \tag{10}$$

$$\rho\frac{\partial v_p}{\partial t} + \rho\dot{p}\frac{\partial v_p}{\partial p} + \left(\frac{p^2 + q_0^2}{p^2 + q_0^2 - q^2}\right)^{1/2}\frac{\partial P}{\partial p} = 0, \tag{11}$$

$$\frac{\partial}{\partial t}\left(e + \frac{1}{2}v_p^2\right) + \dot{p}\frac{\partial}{\partial p}\left(e + \frac{1}{2}v_p^2\right) + \frac{1}{\rho(p^2 + q_0^2 - q^2)}\frac{\partial}{\partial p}[\dot{p}P(p^2 + q_0^2 - q^2)] = \dot{Q}, \tag{12}$$

$$\rho \dot{Q}_C = \frac{1}{p^2 + q_0^2 - q^2}\frac{\partial}{\partial p}\left[(p^2 + q_0^2)\, K\frac{\partial T}{\partial p}\right], \tag{13}$$

$$\rho \dot{Q}_L = -\frac{1}{p^2 + q_0^2 - q^2}\frac{\partial}{\partial p}[(p^2 + q_0^2)^{1/2}(p^2 + q_0^2 - q^2)^{1/2}\,\Phi_p], \tag{14}$$

where $\dot{p} \equiv dp/dt = v_p(p^2 + q_0^2)^{1/2}/(p^2 + q_0^2 - q^2)^{1/2}$.

Unlike the case of spherical coordinates, Eqs. (10)–(14) are still q-dependent, even though we have assumed P, T, etc., to be independent of q. This is due to the fact that the "ellipsoidal symmetry" is only approximate. In other words, if we were to take as initial conditions, temperatures, pressures, etc., that were independent of q, then the solutions of the full two-(or three-)dimensional equations would not remain independent of q for later times. However, the q dependence is only significant in the region $q \approx q_0$ and $|p| \leqslant q_0$. In this region we expect our approximation to be poor anyway since this is initially part of the vacuum-solid interface outside of the focal region, and flow across this interface is being ignored. Consequently, we will concentrate on the regions for which $q^2 \ll q_0^2$, that is, in the vicinity of the z axis. As a result, Eqs. (10)–(14) simplify somewhat:

$$\frac{\partial \rho}{\partial t} + v_p \frac{\partial \rho}{\partial p} + \frac{\rho}{R} \frac{\partial}{\partial p} (R v_p) = 0, \tag{15}$$

$$\rho \frac{\partial v_p}{\partial t} + \rho v_p \frac{\partial v_p}{\partial p} + \frac{\partial P}{\partial p} = 0, \tag{16}$$

$$\frac{\partial}{\partial t} \left(e + \frac{1}{2} v_p^2\right) + v_p \frac{\partial}{\partial p} \left(e + \frac{1}{2} v_p^2\right) + \frac{1}{\rho R} \frac{\partial}{\partial p} (R v_p P) = \dot{Q}, \tag{17}$$

$$\dot{Q}_C = \frac{1}{\rho R} \frac{\partial}{\partial p} \left(RK \frac{\partial T}{\partial p}\right), \tag{18}$$

$$\dot{Q}_L = -\frac{1}{\rho R} \frac{\partial}{\partial p} (R \Phi_p), \tag{19}$$

with

$$R = p^2 + q_0^2, \tag{20}$$

where, now, $v_p = \dot{p}$.

Equations (15)–(19) are very similar to the one-dimensional equations that occur in planar, cylindrical, and spherical symmetry. The only difference is that the factor $R = (p^2 + q_0^2)$ replaces $r^{\delta - 1}$, where $\delta = 1$, 2, or 3, respectively for the three symmetries. Indeed, Eqs. (15)–(19) become equivalent to the spherical ones for $p^2 \gg q_0^2$. Since R always occurs in the combination $R^{-1}\partial(R \cdots)/\partial p$, multiplication of R by a constant leaves the equations unchanged and $(p^2 + q_0^2)$ can be replaced by $(1 + p^2/q_0^2)$. Then, for $p^2 \ll q_0^2$ Eqs. (15)–(19) become equivalent to the corresponding one-dimensional planar equations.

In the process of obtaining Eqs. (15)–(19), our original approximation has been altered slightly. In effect, we are now assuming that the contours of constant temperature, etc., are elliptical in the vicinity of the z axis and that, consequently, Eqs. (15)–(19) describe the plasma behavior in this region. Since we are now trying to match the solution mainly along this axis, we have a certain amount of freedom to vary q_0 to obtain the best match to the correct two-dimensional solution. On physical grounds, however, we would expect q_0 to be approximately equal to r_0, the radius of the focal spot. The full elliptical coordinates are used primarily to extrapolate the solution from the z axis to the remainder of the material for the purposes of calculating neutron production, etc.

Equations (15)–(19) can be readily changed to Lagrangian form with the substitution

$$d/dt = \partial/\partial t + v_p \partial/\partial p. \tag{21}$$

Equation (15) then becomes

$$\frac{d\rho}{dt} + \frac{\rho}{R}\frac{\partial}{\partial p}(Rv_p) = 0. \tag{22}$$

If we define $m(p)$ by

$$m(p) = \int_{-\infty}^{p} R(p')\,\rho(p')\,dp', \tag{23}$$

then, using Eq. (15), it can easily be verified that

$$\frac{dm}{dt} = \frac{\partial m}{dt} + v_p\frac{\partial m}{\partial p} = 0. \tag{24}$$

The quantity $m(p)$ is proportional to the mass to the left of the surface $p =$ constant and inside the surface $q =$ constant $\leqslant q_0$. Equation (24) is then just the conservation of mass: for a point $p(t)$ moving with the fluid, m is a constant.

Consequently, m can be used as a Lagrangian coordinate through the substitution

$$\frac{\partial}{\partial p} = \frac{\partial m}{\partial p}\frac{\partial}{\partial m} = R\rho\frac{\partial}{\partial m}. \tag{25}$$

Utilizing Eqs. (21) and (25) the remaining equations, (16)–(19), become

$$\frac{dv_p}{dt} + R\frac{\partial P}{\partial m} = 0, \tag{26}$$

$$\frac{d}{dt}\left(e + \frac{1}{2}v_p{}^2\right) + \frac{\partial}{\partial m}(Rv_pP) = \dot{Q}, \tag{27}$$

$$\dot{Q}_C = \frac{\partial}{\partial m}\left(RK\frac{\partial T}{\partial p}\right), \tag{28}$$

$$\dot{Q}_L = -\frac{\partial\psi}{\partial m}, \tag{29}$$

where

$$\psi = R\Phi_p. \tag{30}$$

The simplest equation for the propagation of the incoming laser energy is

$$\nabla\cdot\mathbf{\Phi} = -\kappa\,|\,\mathbf{\Phi}\,|, \tag{31}$$

where κ is the absorption coefficient. By assumption Φ_p is the only nonzero component. Utilizing Eq. (30) the propagation equation may be written

$$\frac{\partial\psi_+}{\partial p} = -\kappa\,|\,\psi_+\,|, \tag{32}$$

where now ψ_+ denotes the incoming laser energy. Note that ψ_+ is negative if the laser is incident from the $+p$ direction. If the density exceeds the critical density at some point, the energy is either reflected or "anomalously absorbed" at this point. In the case of reflection, the reflected energy is given by $\psi_- = -\psi_+$ at the critical density. The reflected energy propagates back out of the plasma according to the equation

$$\frac{\partial \psi_-}{\partial p} = -\kappa \mid \psi_- \mid. \tag{33}$$

For the purposes of Eq. (29), the total flux is given by $\psi = \psi_+ + \psi_-$.

4. Discussion

In the preceding section, one-dimensional equations were obtained using an ellipsoidal coordinate system. Similar results can be obtained on a simpler phenomenological basis: We note that for all of the one-dimensional models, the relevant equations are the same except for the factor R which is determined (aside from a multiplicative constant) by the coordinate system. In the laser–plasma problem the geometry in the vicinity of the target surface ($p = 0$) should be essentially planar so that $R \approx 1$ for small p. For distances far from the surface the plasma motion should be similar to that produced by a point source, i.e., spherical. Thus, $R \approx p^2$ (or p^2/q_0^2) for large $\mid p \mid$. A form of R that satisfies these requirements is

$$R = [1 + (p/q_0)^{2n}]^{1/n}. \tag{34}$$

The value of n determines the sharpness of the transition from planar to spherical geometry. We have chosen $n = 1$, which results in a smooth transition.

In a related model, Floux [4] joins together a conical section and a cylindrical section, inside of which the geometries are spherical and planar, respectively. In effect, he has taken $n = \infty$ in Eq. (34) so that $R = 1$ for $p < q_0$ and $R = (p/q_0)^2$ for $p > q_0$. A comparison of the geometries for $n = 1$ and $n = \infty$ is shown in Fig. 3. As can readily be seen, the two models differ mainly in the vicinity of $p = q_0$.

At this time, the best method we have of checking the results of one-dimensional ellipsoidal calculations is to compare them with the results of full two-dimensional calculations. Such comparisons are being made, and the results so far are very encouraging. Comparisons for several typical laser–plasma problems will be published in a forthcoming paper.

In the foregoing section the application of ellipsoidal coordinates to one-dimensional calculations has been discussed. It should be pointed out that an ellipsoidal coordinate system can be used to advantage in two-dimensional calculations as well, especially in Eulerian codes. The essential feature of the ellipsoidal coordinates is that in a laser–plasma problem the density, temperature, etc., should be fairly constant along the ellipses. Consequently, relatively few zones should be needed in the q direction. Indeed, a one-dimensional calculation may be regarded as a limiting case where only one zone in the q direction is used. The

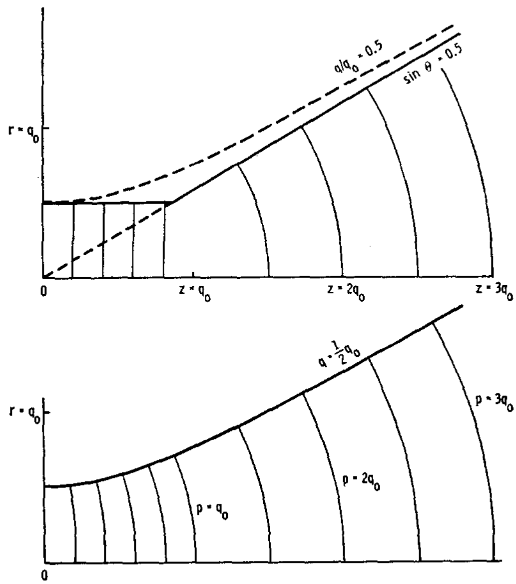

FIG. 3. Comparison of geometries used by Floux [2] (upper) and in this paper (lower).

hydrodynamic equations for the more general two-dimensional case are given in Appendix A. Using the two-dimensional equations, each of the functions may be expanded as a power series in q, for example $\rho = \rho_0 + \rho_1 q + \rho_2 q^2 + \cdots$, where ρ_0, ρ_1, etc. are functions of p and t. From symmetry considerations, all of the scalar functions (ρ, T, etc.) and the p-components of vectors must be even functions of q, while the q components of vectors must be odd functions. This means, for example, that ρ_1, ρ_3, ρ_5, etc. are all zero. By retaining only the zero order terms, Eqs. (15)–(19) are obtained. In some cases it may be necessary to include some higher order terms. At this time no general criterion is available for determining how small the higher order terms are. The best check on the accuracy of the one-dimensional calculation is a comparison with a two dimensional calculation. One such comparison should suffice for each general category of problems.

5. Example

To show the difference between one-dimensional calculations using planar geometry and those using ellipsoidal geometry, we present the results of a sample laser problem in this section. The problem consists of irradiating a deuterium slab with initial density of 0.21 gm/cm^3 by a laser pulse with a total fluence of 3.2×10^4 J/mm^2 in a triangular pulse with a base 0.3 nsec long. This corresponds to 250 J in a 100 μm diameter circle. WOLIP, a one-dimensional, Lagrangian laser–plasma code was used for the calculation. The difference equations used in WOLIP are given in Appendix B. In addition to Eqs. (26)–(33), WOLIP has separate electron and ion temperatures, heat exchange between the electrons and ions, and bremsstrahlung loss from the electrons. An ideal-gas equation of state is used in these examples. WOLIP also calculates the neutrons produced from fusion, however, the energy produced by fusion is ignored. In these calculations anomalous absorption as well as inverse bremsstrahlung absorption is used: any laser energy reaching the critical density is absorbed at that point. This is done to ensure that the same amount of energy is absorbed in each calculation. The temperatures and densities at 0.3 nsec, the end of the laser pulse, are shown in Figs. 4 and 5. The laser is incident from the right; the surface of the solid deuterium was initially at $x = 0.0$. Two sets of curves are shown, one for planar geometry, the other for ellipsoidal geometry with $q_0 = 50\mu$m. Initially the problem is dominated by electron thermal conduction. As can be seen in Fig. 4, a thermal conduction wave has propagated

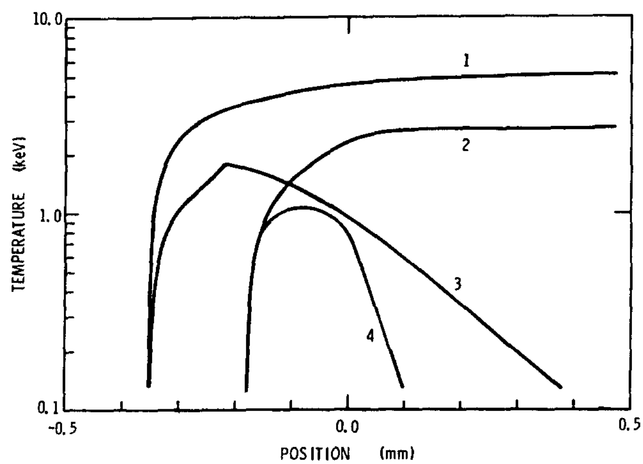

Fig. 4. Temperatures at the end of a 3.2×10^4 J/mm^2, 0.3 nsec laser pulse incident from the right on solid deuterium. Curves 1 and 2 are electron temperature, 3 and 4, ion temperature. Curves 1 and 3 are the result of a calculation using planar geometry, 2 and 4, ellipsoidal geometry with $q_0 = 50$ μm.

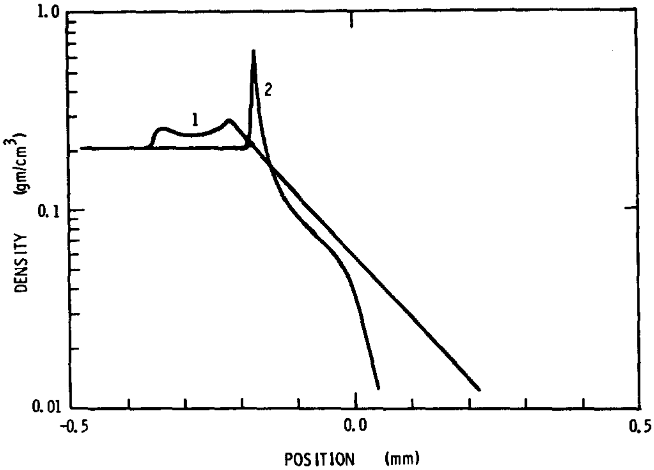

FIG. 5. Densities for the same cases as Fig. 4. Curve 1 is for planar geometry, 2 is for ellipsoidal geometry.

into the solid. Due to lateral energy flow (thermal conduction) in the ellipsoidal case, the temperatures are lower and the thermal conduction wave is moving more slowly. As a result, a shock wave has become fairly well formed, resulting in the large density peak seen in Fig. 5. In the planar case, the thermal conduction wave is just beginning to slow down enough for a shock wave to start forming (the bump in density at -0.35 mm). As the deuterium expands into the vacuum, the density and ion temperature drop off more rapidly in the ellipsoidal case due to lateral expansion. As a result of the lower temperatures, about 1/30 as many neutrons are produced in the ellipsoidal case.

APPENDIX A: HYDRODYNAMICS IN ORTHOGONAL CURVILINEAR COORDINATES

1. *General*

The hydrodynamic equations with which we are concerned may be written in the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{A1}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\, \mathbf{v} + \nabla P = 0, \tag{A2}$$

$$\frac{\partial}{\partial t}\left(e + \frac{1}{2} v^2\right) + \mathbf{v} \cdot \nabla \left(e + \frac{1}{2} v^2\right) + \frac{1}{\rho} \nabla \cdot (P\mathbf{v}) = \dot{Q}, \tag{A3}$$

$$\rho \dot{Q}_C = \nabla \cdot (K \nabla T), \tag{A4}$$

$$\rho \dot{Q}_L = -\nabla \cdot \mathbf{\Phi}. \tag{A5}$$

These equations are identical to Eq. (5)–(9). Anderson, Preiser, and Rubin [5] discuss a conservative form of these equations with $\dot{Q} = 0$ for general curvilinear coordinate systems. The foregoing equations are used here because they can be readily changed to Lagrangian form. A conservative form of the equations can be obtained using the results of this paper and Ref. [5]. The differential operators in orthogonal curvilinear coordinates are given by [6–10]

$$\nabla f = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial f}{\partial x_i}, \tag{A6}$$

$$\nabla \cdot \mathbf{A} = \sum_i \frac{1}{h} \frac{\partial}{\partial x_i} \left(\frac{h A_i}{h_i} \right), \tag{A7}$$

$$[(\mathbf{v} \cdot \nabla) \mathbf{v}]_j = \sum_i \frac{v_i}{h_i} \left[\frac{1}{h_j} \frac{\partial (h_j v_j)}{\partial x_i} - \frac{v_i}{h_j} \frac{\partial h_i}{\partial x_j} \right], \tag{A8}$$

where $\mathbf{u}_i$ is the unit vector along the x_i coordinate line and h_i is a metric coefficient. The length ds of an infinitesimal line segment is given by

$$ds^2 = \sum_i h_i^2 \, dx_i^2. \tag{A9}$$

The unsubscripted h is the product of all the h_i,

$$h = h_1 h_2 h_3 . \tag{A10}$$

Utilizing Eqs. (A6)–(A8) the hydrodynamic equations become

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{1}{h} \frac{\partial}{\partial x_i} \left(\frac{h \rho v_i}{h_i} \right) = 0 \tag{A11}$$

or

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{v_i}{h_i} \frac{\partial \rho}{\partial x_i} + \rho \sum_i \frac{1}{h} \frac{\partial}{\partial x_i} \left(\frac{h v_i}{h_i} \right) = 0, \tag{A12}$$

$$\rho \frac{\partial v_j}{\partial t} + \rho \sum_i \frac{v_i}{h_i h_j} \left[\frac{\partial (h_j v_j)}{\partial x_i} - v_i \frac{\partial h_i}{\partial x_j} \right] + \frac{1}{h_j} \frac{\partial P}{\partial x_j} = 0, \tag{A13}$$

$$\frac{\partial}{\partial t} \left(e + \frac{1}{2} v^2 \right) + \sum_i \frac{v_i}{h_i} \frac{\partial}{\partial x_i} \left(e + \frac{1}{2} v^2 \right) + \frac{1}{\rho} \sum_i \frac{1}{h} \frac{\partial}{\partial x_i} \left(\frac{h P v_i}{h_i} \right) = \dot{Q}, \tag{A14}$$

$$\dot{Q}_C = \frac{1}{\rho}\sum_i \frac{1}{h}\frac{\partial}{\partial x_i}\left(\frac{hK}{h_i^2}\frac{\partial T}{\partial x_i}\right), \tag{A15}$$

$$\dot{Q}_L = -\frac{1}{\rho}\sum_i \frac{1}{h}\frac{\partial}{\partial x_i}\left(\frac{h\Phi_i}{h_i}\right). \tag{A16}$$

The corresponding set of equations in Lagrangian form can be readily obtained by using the relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_i \dot{x}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \sum_i \frac{v_i}{h_i}\frac{\partial}{\partial x_i} \tag{A17}$$

in Eqs. (A12)–(A14).

2. *Ellipsoidal Coordinates*

The relations between the ellipsoidal coordinates p and q and the cylindrical coordinates r and z are, as given by Eqs. (3) and (4),

$$z = \frac{p}{q_0}(q_0^2 - q^2)^{1/2}, \tag{A18}$$

$$r = \frac{q}{q_0}(p^2 + q_0^2)^{1/2}. \tag{A19}$$

The third coordinate in each case is ϕ, the angle around the z axis. The metric coefficients in the two different coordinate systems are related by

$$(\bar{h}_i)^2 = \sum_j \left(\frac{\partial x_j}{\partial \bar{x}_i} h_j\right)^2. \tag{A20}$$

In cylindrical coordinates $h_r = h_z = 1$ and $h_\phi = r$. Using Eqs. (A18)–(A20) the metric coefficients in ellipsoidal coordinates are

$$h_p = \left[\left(\frac{\partial r}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2\right]^{1/2} = \gamma/\alpha, \tag{A21}$$

$$h_q = \left[\left(\frac{\partial r}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2\right]^{1/2} = \gamma/\beta, \tag{A22}$$

$$h_\phi = r = \frac{q\alpha}{q_0}, \tag{A23}$$

$$h = h_p h_q h_\phi = \frac{q\gamma^2}{q_0\beta}, \tag{A24}$$

where

$$\alpha = (p^2 + q_0^2)^{1/2}, \tag{A25}$$

$$\beta = (q_0^2 - q^2)^{1/2}, \tag{A26}$$

$$\gamma = (p^2 + q_0^2 - q^2)^{1/2}. \tag{A27}$$

Since we are concerned here with axially symmetric problems, we will omit all of the derivatives with respect to ϕ and assume $v_\phi = 0$. As a consequence, the hydrodynamic equations (A11), (A13)–(A16) become

$$\frac{\partial \rho}{\partial t} + \frac{1}{\gamma^2}\frac{\partial}{\partial p}(\alpha\gamma\rho v_p) + \frac{\beta}{q\gamma^2}\frac{\partial}{\partial q}(\gamma q \rho v_q) = 0, \tag{A28}$$

$$\rho\frac{\partial v_p}{\partial t} + \frac{\alpha}{\gamma}\rho v_p\frac{\partial v_p}{\partial p} + \frac{\beta}{\gamma^2}\rho v_q\frac{\partial}{\partial q}(\gamma v_p) - \frac{p\alpha}{\gamma^3}\rho v_q^2 + \frac{\alpha}{\gamma}\frac{\partial P}{\partial p} = 0, \tag{A29}$$

$$\rho\frac{\partial v_q}{\partial t} + \frac{\beta}{\gamma}\rho v_q\frac{\partial v_q}{\partial q} + \frac{\alpha}{\gamma^2}\rho v_p\frac{\partial}{\partial p}(\gamma v_q) + \frac{q\beta}{\gamma^3}\rho v_p^2 + \frac{\beta}{\gamma}\frac{\partial P}{\partial q} = 0, \tag{A30}$$

$$\frac{\partial}{\partial t}\left(e + \frac{v_p^2 + v_q^2}{2}\right) + \frac{\alpha}{\gamma}v_p\frac{\partial}{\partial p}\left(e + \frac{v_p^2 + v^2_q}{2}\right) + \frac{\beta}{\gamma}v_q\frac{\partial}{\partial q}\left(e + \frac{v_p^2 + v_q^2}{2}\right)$$

$$+ \frac{1}{\rho\gamma^2}\frac{\partial}{\partial p}(\alpha\gamma P v_p) + \frac{\beta}{\rho q\gamma^2}\frac{\partial}{\partial q}(q\gamma P v_q) = \dot{Q}, \tag{A31}$$

$$\dot{Q}_C = \frac{1}{\rho\gamma^2}\left[\frac{\partial}{\partial p}\left(\alpha^2 K\frac{\partial T}{\partial p}\right) + \frac{\beta}{q}\frac{\partial}{\partial q}\left(q\beta K\frac{\partial T}{\partial q}\right)\right], \tag{A32}$$

$$\dot{Q}_L = -\frac{1}{\rho\gamma^2}\left[\frac{\partial}{\partial p}(\alpha\gamma\Phi_p) + \frac{\beta}{q}\frac{\partial}{\partial q}(q\gamma\Phi_q)\right]. \tag{A33}$$

Finally we might remark that we could replace the coordinate q by the angular coordinate θ which is defined by

$$\sin\theta \equiv q/q_0 . \tag{A34}$$

The resulting set of coordinates are related to the cylindrical coordinates r, z by the relations

$$z = p\cos\theta, \tag{A35}$$

$$r = (p^2 + q_0^2)^{1/2}\sin\theta. \tag{A36}$$

If we now allow the range of θ to exceed $\pi/2$, the coordinates p and θ are continuous everywhere, removing the discontinuity at $q = q_0$ referred to in Section 2. The hydrodynamic equations can be obtained through the same process as was used to obtain Eqs. (A28)–(A33), or more simply by using Eq. (A34) and the relation

$$\beta \frac{\partial}{\partial q} = \frac{\partial}{\partial \theta}. \tag{A37}$$

As can be seen from Eqs. (A35) and (A36) spherical coordinates can then be obtained as a special case by setting $q_0 = 0$.

APPENDIX B: WOLIP EQUATIONS

WOLIP is a one-dimensional Lagrangian hydrocode written primarily for the study of laser induced plasmas. It has separate ion and electron temperatures, heat exchange between ions and electrons, ion and electron thermal conductivity, bremsstrahlung loss from the electrons, and inverse bremstrahlung laser absorption. WOLIP has a provision for using anomalous laser absorption wherein a specified fraction of the laser energy that reaches the critical density is absorbed at that point; the remainder is reflected back out through the plasma. In addition, WOLIP calculates the neutrons produced in fusion reactions. For the calculations of interest here, the amount of fusion energy produced was negligible and was consequently ignored in the energy balance equation. The equations used by WOLIP are similar to those used in other laser-plasma codes [11]; we include them here for completeness.

In this section, we will denote the spatial variable in all geometries by r. The "radial factor" R is then $R = 1$ for planar geometry, $R = 2\pi r$ for cylindrical geometry, $4\pi r^2$ for spherical geometry, and $2\pi(r^2 + q_0^2)$ for ellipsoidal geometry. The factors 2π and 4π are used so that, for example, in spherical geometry ψ is the total laser power rather than the power per steradian.

In a Lagranigan code mass is automatically conserved by assigning a fixed amount of mass to each zone, denoted by $\Delta m_{j+\frac{1}{2}}$. We denote quantities at the zone boundries with the subscript j and quantities at the zone centers with the subscript $j + \frac{1}{2}$. Similarly, quantities at time t^n have the superscript n, quantities at intermediate times have the superscript $n + \frac{1}{2}$. We denote spatial differences with a Δ and temporal differences with a δ, thus,

$$\Delta a_j^n \equiv a_{j+\frac{1}{2}}^n - a_{j-\frac{1}{2}}^n, \tag{B1}$$

$$\delta a_j^n \equiv a_j^{n+\frac{1}{2}} - a_j^{n-\frac{1}{2}}. \tag{B2}$$

Although n and j are usually taken to be integers, Eqs. (B1) and (B2) hold for both integral and half-integral n and j. In addition to Δm, the principal variables in WOLIP are t^n, r_j^n, $v_j^{n+\frac{1}{2}}$, $V_{j+\frac{1}{2}}^n$, $T_{i,j+\frac{1}{2}}^n$, and $T_{e,j+\frac{1}{2}}^n$. The last three variables are the specific volume ($V = 1/\rho$) and the ion and electron temperatures, respectively. The pressures P_i and P_e, and the internal energies e_i and e_e are determined by the equations of state, which for the examples considered in this paper are fully ionized, ideal gas equations of state

$$P_i = R_0 T_i / AV, \tag{B3}$$

$$P_e = Z R_0 T_e / AV \tag{B4}$$

$$e_i = \tfrac{3}{2} R_0 T_i / A, \tag{B5}$$

$$e_e = \tfrac{3}{2} Z R_0 T_e / A, \tag{B6}$$

where R_0 is the gas constant, and A and Z are the atomic weight and atomic numbers, respectively. When full ionization is not appropriate, Z is replaced by $\bar{Z}$, the average charge state. Similarly, for compounds or other mixtures of nuclei, A is replaced by $\bar{A}$.

The momentum equation (equation of motion) is

$$\frac{dv}{dt} = -R \frac{\partial}{\partial m} (P + q_v), \tag{B7}$$

where q_v is the VonNeuman artifical viscosity [12] and $P \equiv P_i + P_e$. The difference form of this equation used in WOLIP to obtain $v_j^{n+\frac{1}{2}}$ is

$$\delta v_j^n = -R_j^n (\Delta P_j^n + \Delta q_{v,j}^{n-\frac{1}{2}})\, \delta t^n / \Delta m_j, \tag{B8}$$

with

$$\delta t^n = \tfrac{1}{2}(\delta t^{n+\frac{1}{2}} + \delta t^{n-\frac{1}{2}}), \tag{B9}$$

$$\Delta m_j = \tfrac{1}{2}(\Delta m_{j+\frac{1}{2}} + \Delta m_{j-\frac{1}{2}}). \tag{B10}$$

The form of the artificial viscosity used in WOLIP is

$$q_{v,j+\frac{1}{2}}^{n-\frac{1}{2}} = 2(\Delta v_{j+\frac{1}{2}}^{n-\frac{1}{2}}) / V_{j+\frac{1}{2}}^{n-\frac{1}{2}}, \tag{B11}$$

where

$$V_{j+\frac{1}{2}}^{n-\frac{1}{2}} = \tfrac{1}{2}(V_{j+\frac{1}{2}}^{n} + V_{j+\frac{1}{2}}^{n-1}). \tag{B12}$$

However, if either $\Delta v_{j+\frac{1}{2}}^{n-\frac{1}{2}}$ or $\delta V_{j+\frac{1}{2}}^{n-\frac{1}{2}}$ is positive (expansion) q_v is set equal to zero. Using the results of Eq. (B8), r_j is advanced to t^{n+1}

$$\delta r_j^{n+\frac{1}{2}} = v_j^{n+\frac{1}{2}}\, \delta t^{n+\frac{1}{2}}. \tag{B13}$$

The new specific volume is then obtained with

$$V_{j+\frac{1}{2}}^{n+1} = \Delta U_{j+\frac{1}{2}}^{n+1} / \Delta m_{j+\frac{1}{2}} . \tag{B14}$$

The factor ΔU is obtained from the analytic evaluation of the integral

$$U_j = \int^{r_j} R dr. \tag{B15}$$

For example, for ellipsoidal geometry

$$U_j = 2\pi r_j (q_0^2 + r_j^2/3), \tag{B16}$$

and ΔU is obtained using Eq. (B1).

The energy conservation equation used in WOLIP is not Eq. (27) but may be obtained from it by using Eqs. (22), (25), (26), and the thermodynamic relations

$$\frac{de}{dt} = \frac{dT}{dt}\left[\frac{\partial e}{\partial T}\right]_V + \frac{dV}{dt}\left[\frac{\partial e}{\partial V}\right]_T, \tag{B17}$$

$$\left[\frac{\partial e}{\partial V}\right]_T = T\left[\frac{\partial P}{\partial T}\right]_V - P. \tag{B18}$$

The equation used in WOLIP is

$$C\frac{dT}{dt} + P_T T \dot{V} = \dot{Q}, \tag{B19}$$

where $C = [\partial e/\partial T]_V$, the specific heat at constant volume, $P_T = [\partial P/\partial T]_V$, and $\dot{V} = dV/dt$. The first two of these derivatives are determined from the equations of state. For the case of an ideal gas used here, they take the forms

$$P_{Ti} = R_0/VA \tag{B20}$$

$$P_{Te} = ZR_0/VA, \tag{B21}$$

$$C_i = \tfrac{3}{2}R_0/A \tag{B22}$$

$$C_e = \tfrac{3}{2}ZR_0/A. \tag{B23}$$

Since WOLIP has separate ion and electron temperatures, Eq. (B19) is used twice, once for the ions and once for the electrons,

$$C_i\frac{dT_i}{dt} + P_{Ti}T_i\dot{V} = \dot{Q}_i , \tag{B24}$$

$$C_e\frac{dT_e}{dt} + P_{Te}T_e\dot{V} = \dot{Q}_e . \tag{B25}$$

These two equations may be combined into a single matrix equation having the same form as Eq. (B19) except that T and $\dot{Q}$ are two-element column matrices

$$T = \begin{bmatrix} T_i \\ T_e \end{bmatrix}, \tag{B26}$$

$$\dot{Q} = \begin{bmatrix} \dot{Q}_i \\ \dot{Q}_e \end{bmatrix}, \tag{B27}$$

and C and P_T are square diagonal matrices

$$C = \begin{bmatrix} C_i & 0 \\ 0 & C_e \end{bmatrix}, \tag{B28}$$

$$P_T = \begin{bmatrix} P_{Ti} & 0 \\ 0 & P_{Te} \end{bmatrix}. \tag{B29}$$

The source terms are given by

$$\dot{Q}_i = \dot{Q}_{Ci} + \dot{Q}_X - q_v \dot{V}, \tag{B30}$$

$$\dot{Q}_e = \dot{Q}_{Ce} - \dot{Q}_X + \dot{Q}_R + \dot{Q}_L. \tag{B31}$$

The conductivity terms $\dot{Q}_{Ci}$ and $\dot{Q}_{Ce}$ are given in matrix form by

$$\dot{Q}_C = \frac{\partial}{\partial m}\left(RK\frac{\partial T}{\partial r}\right), \tag{B32}$$

where K is a square diagonal matrix

$$K = \begin{bmatrix} K_i & 0 \\ 0 & K_e \end{bmatrix}. \tag{B33}$$

The heat exchange and radiation loss terms $\dot{Q}_X$ and $\dot{Q}_R$ are given by

$$\dot{Q}_X = 2\omega_X(T_e - T_i), \tag{B34}$$

$$\dot{Q}_R = 2\omega_R T_e. \tag{B35}$$

These are combined into one matrix term, $2\Omega T$, by defining the square matrix

$$\Omega = \begin{bmatrix} -\omega_X & \omega_X \\ \omega_X & -\omega_X - \omega_R \end{bmatrix} \tag{B36}$$

The term $-q_v \dot{V}$ in Eq. (B30) accounts for the shock heating due to ion viscosity.

The laser absorption term $\dot{Q}_L$ is given by Eqs. (29)–(33). The entire source term is then written as

$$\dot{Q} = \dot{Q}_C + 2\Omega T + S, \tag{B37}$$

where

$$S = \begin{bmatrix} -q_v \dot{V} \\ \dot{Q}_L \end{bmatrix}. \tag{B38}$$

Equations (B19) and (B37) lead to the difference equation used in WOLIP to obtain $T_{j+\frac{1}{2}}^{n+1}$

$$\begin{aligned} C_{j+\frac{1}{2}}^{n+\frac{1}{2}}\, \delta T_{j+\frac{1}{2}}^{n+\frac{1}{2}}/\delta t^{n+\frac{1}{2}} &= -P_{T,j+\frac{1}{2}}^{n+\frac{1}{2}} T_{j+\frac{1}{2}}^{n+\frac{1}{2}} \dot{V}_{j+\frac{1}{2}}^{n+\frac{1}{2}} \\ &\quad + (R_{j+1}^{n+\frac{1}{2}} K_{j+1}^{n+\frac{1}{2}}\, \Delta T_{j+1}^{n+\frac{1}{2}}/\Delta r_{j+1}^{n+\frac{1}{2}} - R_j^{n+\frac{1}{2}} K_j^{n+\frac{1}{2}}\, \Delta T_j^{n+\frac{1}{2}}/\Delta r_j^{n+\frac{1}{2}})/\Delta m_{j+\frac{1}{2}} \\ &\quad + 2\Omega_{j+\frac{1}{2}}^{n+\frac{1}{2}} T_{j+\frac{1}{2}}^{n+\frac{1}{2}} + S_{j+\frac{1}{2}}^{n+\frac{1}{2}}, \end{aligned} \tag{B39}$$

where

$$T_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \tfrac{1}{2}(T_{j+\frac{1}{2}}^{n+1} + T_{j+\frac{1}{2}}^{n}), \tag{B40}$$

$$\dot{V}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \delta V_{j+\frac{1}{2}}^{n+\frac{1}{2}}/\delta t^{n+\frac{1}{2}}, \tag{B41}$$

$$\Delta r_j^{n+\frac{1}{2}} = \tfrac{1}{4}(\Delta r_{j+\frac{1}{2}}^{n+1} + \Delta r_{j+\frac{1}{2}}^{n} + \Delta r_{j-\frac{1}{2}}^{n+\frac{1}{2}} + \Delta r_{j-\frac{1}{2}}^{n}), \tag{B42}$$

and $R_j^{n+\frac{1}{2}}$ is evaluated at $r_j^{n+\frac{1}{2}} = \frac{1}{2}(r_j^{n+1} + r_j^{n})$. The coefficients K_i, K_e, ω_R, ω_X, and κ are obtained from Spitzer [13]. They are functions of $T_i^{n+\frac{1}{2}}$ and $T_e^{n+\frac{1}{2}}$ which are obtained by linear extrapolation of T_i^{n-1}, T_i^{n}, T_e^{n-1}, and T_e^{n}. Equation (B39) is an implicit equation since the right hand side contains $T_{j-\frac{1}{2}}^{n+1}$, $T_{j+\frac{1}{2}}^{n+1}$, and $T_{j+\frac{3}{2}}^{n+1}$. It is solved by the usual method for tridiagonal systems, modified to take account of the fact that the elements of the tridiagonal matrix are themselves 2×2 matrices [11].

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